Energy of embedded surfaces invariant under Mœbius transformations, addendum

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Abstract: In a previous preprint we defined an energy associated to every embedding of a surface in R^n or S^n. This energy is invariant under Mœbius transformation and the "round" sphere is its only absolute minimum.Here we sketch a proof of the compactness property for a variant of it. The details will appear elsewhere.

Given an embedding f of S^1 into R^3 or S^3, it is possible to associate to it an "energy" defined as

$$E_0(f) = \int\int_{S^1 \times S^1} \Big(\frac{1}{|x-y|^2} - \frac{1}{|f(x)-f(y)|^2} \Big) dx dy.$$

where x and y are two points on the unit circle, $|x-y|$ is the length of the cord and $|f(x) - f(y)|$ is the euclidean distance, in the case the ambient space is R^3, or the cord distance in the case of S^3. The term $\frac{1}{|x-y|^2}$ is necessary to make the integral convergent and is usually called regularization in the literature. For a link e.g. $f \sqcup g$ one adds to $E_0(f)$ and $E_0(g)$ an uregularized energy:

$$E_u(f;g) = \int\int_{S^1 \times S^1} \frac{1}{|f(x)-g(y)|^2} dx dy.$$

where x and y lie on different circles. In [O'] O'Hara proved that if one adds to $E_0(f)$ a term involving the geodesic curvature $\kappa(x)$ of the knot and defines a new energy

$$E_\lambda(f) = E_0 + \lambda \int_{S^1} (\kappa(x))^2 dx$$

with $\lambda > 0$, the new functional has the compactness property:

The number of knot types that can be realized by an embedding f with $E_\lambda(f) < K$ is less than $N(K)$ where $N(K)$ is some finite function of K.

Freedman, He and Wang discovered that $E_0(f)$ is invariant under Mœbius transformations of S^3 and where able to prove, among other things, the two foundamental properties:

1) The "round" circle is the only absolute minimum of $E_0(f)$
2) $E_0(f)$ has the compactness property.

There has been some effort to find an analogue of the energy for embeddings of S^2 and recently Knuser and Sullivan [KS] and Auckly and Sadun [AS] have proposed several candidates. The energy should satisfy the following axioms, see [AS]:

1) $E_0(f)$ is invariant under Mœbius transformations
2) $E_0(f)$ is bounded below.
3) $E_0(f)$ blows up when the embedding approaches a non-injective map.

4) Given two embeddings f and g, for every ϵ there is an embedding h in the isotopy class of the connnected sum $f \# g$ such that $E_0(h) \leq E_0(f) + E_0(g) + \epsilon$.

5) There exists an unregularized energy for pairs of disjoint embeddings, $E_u(f; g)$, such that, if we put $E(f \sqcup g) = E_0(f) + E_0(g) + E_u(f; g)$, $E(f \sqcup g)$ is invariant for Mœbius transformations.

The Energy of [KS] does not satisfy 5, that of [AS] is not known to satisfy 2, in [D] we proposed an energy for embeddings of S^2 into R^n or S^n that satisfies all the five axioms.

Moreover we showed that the round sphere is the only absolute minimum for $E_0(f)$. Here we prove that, if we add to $E_0(f)$ an additional term similar to the one of [O'], our new functional has the compactness property for embeddings into R^4 or S^4, which is the only meaningful case.

Our definition can be easily extended to give an energy for embeddings of surfaces of any genus. There is some evidence that, in the case of tori, the minimum is given by the Clifford torus.

Definition of the Energy

The energy $E_0(f)$ was defined in [D] as:

$$\int \int_{S^2 \times S^2} \left(\frac{1}{|x-y|^2} - \frac{d_f(x) d_f(y)}{|f(x)-f(y)|^2} \right)^2 dx dy.$$

and there we sketched the proof that it satisfies the five properties quoted before and also that $E_0(f) = 0$ if and only if f is the embedding of the round S^2. Here we shall define a new energy:

$$E_\lambda(f) = E_0(f) + \lambda \left(\int_{f(S^2)} (|H|^2 - 2K)^p dvol \right)^{1/p} Vol(f(S^2))^{1/q}$$

where H is the mean curvature vector of the surface in R^n, $\lambda > 0$, $1/p + 1/q = 1$ and $p > 1$. Remark that the second term is invariant for dilatations but not for Mœbius tranformations anymore. However by the Schwarz inequality:

$$\left(\int_{f(S^2)} (|H|^2 - 2K)^p dvol \right)^{1/p} Vol(f(S^2))^{1/q} \geq \int_{f(S^2)} (|H|^2 - 2K) dvol$$

and the last term is invariant under Mœbius tranformations, so $E_\lambda(\mu \circ f)$ is bounded below when μ varies over all the Mœbius transformations which do not send a point of $f(S^2)$ to infinity. Now we sketch the proof of the compactness property:

Theorem: Let f be an embedding of S^2 into R^4 with $E_\lambda(f) \leq C$, then f can belong to a finite number $N(C)$ of isotopy classes of knots only.

Proof: We break the proof in two lemmas. In the first we will use the second term of the energy to get some local regularity, roughly Lemma 1 shows that there is a uniform length scale in which the surface is very close in the C^1 norm to

its tangent plane.The second lemma uses a theorem of Teichmueller on conformal moduli of planar domains to show that an upper bound on $E_\lambda(f)$ gives a lower bound for the Kuiper selfdistance of the image. The theorem is a consequence of this last fact.

Lemma 1 :Given any $\delta > 0$ we can cover $f(S^2)$ with $N(\delta, \lambda, C)$ 4-balls B_i so that for any i and any $a \in B_i \cap f(S^2)$, if $Tf(S^2)|_a$ denotes the tangent plane in a, the ortogonal projection $\pi : B_i \mapsto Tf(S^2)|_a$ is a diffeomorfism with biLipschitz constant $1 + \delta/2$, in particular it is a $1 + \delta$ quasiconformal homeomorphism. *We can also assume that the center of B_i is on $f(S^2)$.*

Proof: Since $E_0(f)$ is always nonnegative we must have:

$$\left(\int_{f(S^2)} (|H|^2 - 2K)^p dvol \right)^{1/p} \leq \frac{C}{\lambda Vol(f(S^2))^{1/q}}$$

now it is easy to see that the 2p-th powers of the components of the second foundamental form of the surface are bounded by $(|H|^2 - 2K)^p$ and so the left hand side of the inequality is the the square of the L^{2p} norm of the second foundamental form. The latter is also the derivative of the Gauss map T from the surface into the Grassmannian of two planes in R^4, it follows that the inequality above gives a bound on the $L^{1,2p}$ norm squared of the Gauss map. We use the Sobolev embedding of $L^{1,2p}$ into $C^{1/q}$, the space of Hœlder functions of exponent $1/q$, to show that for any $\epsilon > 0$ we have:

$$|T(a) - T(b)| \leq A \left(\frac{C}{\lambda (Vol(f(S^2)))^{1/q}} \right)^{1/2} |a - b|^{1/q}$$

here a and b are points on the surface, the vertical bars denote the distances in the Grassmannian and in the surface and A is the constant of the Sobolev embedding. This estimate implies the lemma by an easy gemetric argument.Q.E.D.

In particular this tells us that we can cover $f(S^2)$ with a finite number (depending only on C) of balls in such a way that in each of these balls the surface can be approximated by a disjoint union of planes, we will call them sheets.

Lemma 2 : $E_\lambda(f) < C$ implies that, modulo a refinement of the covering which will increase the number of balls by a function of C only, we can assume that each ball contains only one sheet.

Proof: Let a be the centre of a ball B_i and r be its radius.Let ϵ be the distance between a and a sheet in B_i not containing it, if it exists. We shall prove that if ϵ is too small, the energy cannot be less than C. Let b be the point realizing the distance ϵ and let U_a and U_b be the sheets containing a and b respectively; if ϵ is small enough we can assume that the inner radius of both sheets is at least $r/2$, say. Let V_a and V_b be smaller disks in the induced metric on the sheets whose radius is mr where $m < 1/2$ will be determinated later.We want to prove that the integral:

$$\int_{x \in f^{-1}(V_a)} \int_{y \in f^{-1}(V_b)} \left(\frac{1}{|x - y|^2} - \frac{d_f(x) d_f(y)}{|f(x) - f(y)|^2} \right)^2 dx dy.$$

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goes to infinity when ϵ goes to zero. First of all by the triangle inequality and substitution of variables we have:

$$\left| \int_{x \in f^{-1}(V_a)} \int_{y \in f^{-1}(V_b)} \left(\frac{1}{|x-y|^2} - \frac{d_f(x)d_f(y)}{|f(x)-f(y)|^2} \right)^2 dxdy \right|^{1/2} \geq$$

$$\geq \left| \int_{V_a} \int_{V_b} \frac{1}{|\xi-\eta|^4} d\xi d\eta \right|^{1/2} - \left| \int_{f^{-1}(V_a)} \int_{f^{-1}(V_b)} \frac{1}{|x-y|^4} dxdy \right|^{1/2}$$

where $\xi = f(x)$ and $\eta = f(y)$.

The proof of Lemma 1 shows that the first integral in the right hand side is close to the integral over the projection of V_a and V_b onto their tangent planes at a and b respectively. The latter is greater than the integral over two planar disks of radius $mr/(1+\delta)$ embedded in R^4 at Hausdorff distance less than ϵ. Here the radius of the disks does not depend on ϵ. The evaluation of this integral and the proof that it goes to infinity when $\epsilon \to 0$ is an exercice in calculus which we willingly leave to the reader.

Now we prove that the second integral is bounded. First remark that the $1+\delta$ quasi-conformality of the projection onto the tangent spaces implies that the annuli $U_a - V_a$ and $U_b - V_b$ have conformal modulus very close to $log(2/m)$, and that, since f is conformal, the same is true of their preimages in S^2.

Modulo a Mœbius transformation in the domain we can assume that $f(0) = a$ and $f(\infty) = b$. Now if m is chosen small enough a theorem of Teichmueller on extremal domains (see [LV], p.57) says that

$$\frac{inf_{x \in S^2 - f^{-1}(U_a)}|a-x|}{sup_{y \in f^{-1}(V_a)}|a-y|} \geq D(m)$$

where as usual the vertical bars denote the spherical distance in S^2. Note that D(m) does not depend on ϵ. The same estimate holds for $f^{-1}(U_b)$ and it is easy to see that these two together imply that the integral $\int_{f^{-1}(V_a)} \int_{f^{-1}(V_b)} \frac{1}{|x-y|^4} dxdy$ is bounded independently of ϵ. This ends the proof of Lemma 2 and gives the compactness property. Q.E.D.

References

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